

Control Systems I

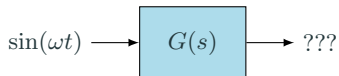
Frequency Response

Colin Jones

Laboratoire d'Automatique

System Frequency Response

Describe the behaviour of the system by how it responds to sinusoidal inputs



Key take-home:

- The output is a sinusoid of a same frequency with a phase shift of $\angle G(j\omega)$ and a magnitude of $|G(j\omega)|$

Why important?

- Very useful method to experimentally capture the dynamics of a system
- Common control objectives expressed in terms of frequency requirements
- Can determine closed-loop behaviour, from open-loop frequency response

Suppose we have a system with a transfer function $G(s)$, and we drive it with the input signal $u(t) = \sin(\omega t)$.

To make things simple, drive the system with the complex signal:

$$u(t) := e^{j\omega t}$$

¹Assume for now that G only has simple poles that are all different from $j\omega$

Suppose we have a system with a transfer function $G(s)$, and we drive it with the input signal $u(t) = \sin(\omega t)$.

To make things simple, drive the system with the complex signal:

$$u(t) := e^{j\omega t}$$

The output is

$$\begin{aligned} Y(s) &= G(s)U(s) \\ &= G(s) \frac{1}{s - j\omega} \\ &= c_0 + \frac{c_1}{s - p_1} + \frac{c_2}{s - p_2} + \cdots + \frac{c_n}{s - p_n} + \frac{c}{s - j\omega} \end{aligned}$$

where $\{p_i\}$ are the unique, simple poles of $G(s)$ ¹

¹Assume for now that G only has simple poles that are all different from $j\omega$

Computing the Steady-State Response

The response to $e^{j\omega t}$ is

$$y(t) = c_0\delta(t) + c_1e^{p_1t} + \cdots + c_ne^{p_nt} + ce^{j\omega t}$$

If $G(s)$ is stable, then for sufficiently large t , this will tend to

$$y(t) = ce^{j\omega t}$$

Computing the Steady-State Response

The response to $e^{j\omega t}$ is

$$y(t) = c_0\delta(t) + c_1e^{p_1t} + \cdots + c_ne^{p_nt} + ce^{j\omega t}$$

If $G(s)$ is stable, then for sufficiently large t , this will tend to

$$y(t) = ce^{j\omega t}$$

Compute c in the standard fashion:

$$\begin{aligned} c &= \lim_{s \rightarrow j\omega} (s - j\omega)G(s)U(s) \\ &= \lim_{s \rightarrow j\omega} (s - j\omega)G(s) \frac{1}{s - j\omega} \\ &= G(j\omega) \end{aligned}$$

Steady-State Response

Use superposition to get the response to $u(t) = \sin(\omega t)$

$$u(t) = \sin(\omega t) = \frac{1}{2j} \left(e^{j\omega t} - e^{-j\omega t} \right)$$

For large t we have that

$$\begin{aligned} y(t) &= \frac{1}{2j} \left(G(j\omega)e^{j\omega t} - G(-j\omega)e^{-j\omega t} \right) \\ &= \frac{|G(j\omega)|}{2j} \left(e^{j\angle G(j\omega)} e^{j\omega t} - e^{-j\angle G(j\omega)} e^{-j\omega t} \right) \\ &= \frac{|G(j\omega)|}{2j} \left(e^{j(\angle G(j\omega) + \omega t)} - e^{-j(\angle G(j\omega) + \omega t)} \right) \\ &= |G(j\omega)| \sin(\omega t + \angle G(j\omega)) \end{aligned}$$

Steady-State Response

Use superposition to get the response to $u(t) = \sin(\omega t)$

$$u(t) = \sin(\omega t) = \frac{1}{2j} \left(e^{j\omega t} - e^{-j\omega t} \right)$$

For large t we have that

$$\begin{aligned} y(t) &= \frac{1}{2j} \left(G(j\omega)e^{j\omega t} - G(-j\omega)e^{-j\omega t} \right) \\ &= \frac{|G(j\omega)|}{2j} \left(e^{j\angle G(j\omega)} e^{j\omega t} - e^{-j\angle G(j\omega)} e^{-j\omega t} \right) \\ &= \frac{|G(j\omega)|}{2j} \left(e^{j(\angle G(j\omega) + \omega t)} - e^{-j(\angle G(j\omega) + \omega t)} \right) \\ &= |G(j\omega)| \sin(\omega t + \angle G(j\omega)) \end{aligned}$$

If the input is a sinusoid at frequency ω , the output is a sinusoid at the same frequency, with the magnitude scaled by $|G(j\omega)|$ and the phase shifted by $\angle G(j\omega)$.

Example

Consider the system $\ddot{y}(t) + 1.1\dot{y}(t) + 0.1y(t) = u(t)$ driven by $u(t) = \sin(1.3t)$

¹Recall that $B \cos(\omega_0 t) + C \sin(\omega_0 t) = A \sin(\omega_0 t + \phi)$, where $\phi = \tan^{-1} \left(\frac{C}{B} \right)$ and $A = \sqrt{B^2 + C^2}$

Example

Consider the system $\ddot{y}(t) + 1.1\dot{y}(t) + 0.1y(t) = u(t)$ driven by $u(t) = \sin(1.3t)$

The output is

$$\begin{aligned} Y(s) = G(s)U(s) &= \frac{1}{(s+1)(s+0.1)} \cdot \frac{1.3}{s^2 + 1.3^2} \\ &= \frac{-0.54}{s+1} + \frac{0.85}{s+0.1} - \frac{0.31s + 0.45}{s^2 + 1.3^2} \end{aligned}$$

Taking the inverse transform we get the time response

$$\begin{aligned} y(t) &= -0.54e^{-t} + 0.85e^{-0.1t} - 0.31 \cos(1.3t) - 0.45 \sin(1.3t) \\ &= -0.54e^{-t} + 0.85e^{-0.1t} + 0.47 \sin(1.3t - 2.41) \end{aligned}$$

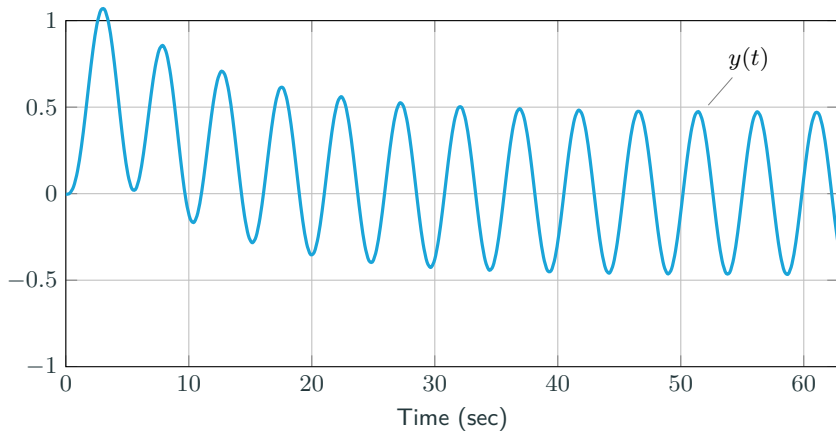
Note that

$$G(j1.3) = -0.35 - 0.31j = 0.47e^{-2.41j}$$

¹Recall that $B \cos(\omega_0 t) + C \sin(\omega_0 t) = A \sin(\omega_0 t + \phi)$, where $\phi = \tan^{-1} \left(\frac{C}{B} \right)$ and $A = \sqrt{B^2 + C^2}$

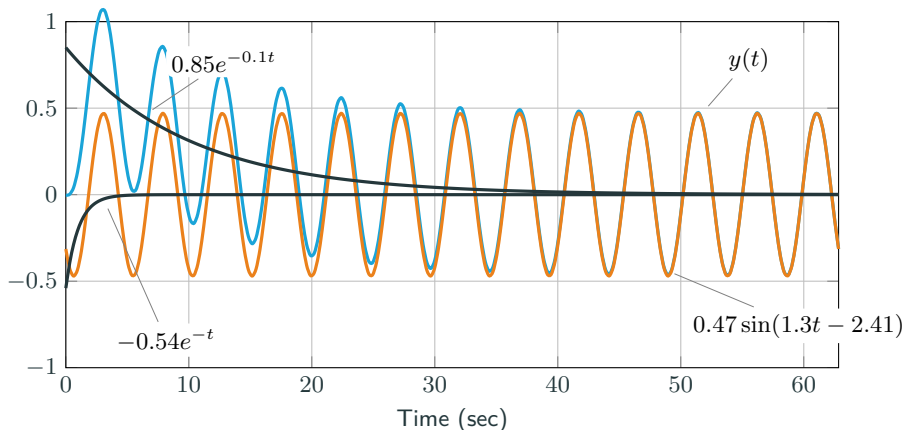
Example

$$y(t) = -0.54e^{-t} + 0.85e^{-0.1t} + 0.47 \sin(1.3t - 2.41)$$



Example

$$y(t) = -0.54e^{-t} + 0.85e^{-0.1t} + 0.47 \sin(1.3t - 2.41)$$



Only the sinusoid counts in the long-term.

Initial response is called the **transient response**, the long-term is called the **steady-state response**

Frequency Response

The **frequency response** of the transfer function $G(s)$ is the function $G(j\omega)$

- If any of the poles are unstable, $\text{Re } p_i > 0$, then the transient will not die-out

$$\lim_{t \rightarrow \infty} |e^{-p_i t}| = \infty$$

- If any poles are on the imaginary axis, $p_i = j\omega_i$, then their response will not die out either, but will tend to a sinusoid with a frequency of ω_i .

In these cases, the frequency response is still well-defined, ***but it no longer defines the steady-state response!***

Complex Sinusoidal Inputs

Phase shifted input

$$u(t) = \sin(\omega t + \phi)$$

What is $y(t)$?

Complex Sinusoidal Inputs

Phase shifted input

$$u(t) = \sin(\omega t + \phi)$$

What is $y(t)$?

Change of variables $\tau = t + \frac{\phi}{\omega}$

$$u(\tau) = \sin(\omega \tau)$$

$$\Rightarrow y(\tau) = |G(j\omega)| \sin(\omega \tau + \angle G(j\omega))$$

Undoing change of variables gives

$$y(t) = |G(j\omega)| \sin(\omega t + \phi + \angle G(j\omega))$$

Complex Sinusoidal Inputs

Phase shifted input

$$u(t) = \sin(\omega t + \phi)$$

What is $y(t)$?

Change of variables $\tau = t + \frac{\phi}{\omega}$

$$u(\tau) = \sin(\omega \tau)$$

$$\Rightarrow y(\tau) = |G(j\omega)| \sin(\omega \tau + \angle G(j\omega))$$

Undoing change of variables gives

$$y(t) = |G(j\omega)| \sin(\omega t + \phi + \angle G(j\omega))$$

Idea: Which time is 'zero' doesn't matter when we're talking about a signal running infinitely far into the past and into the future

Example

$$u(t) = \sin(2\pi/3t) + 1.2 \sin(\pi t + 0.3) + 0.8 \sin(2\pi/5t + 0.4)$$

What's the steady-output of a system with transfer function $G(s)$, if the system has all poles in the left half plane?

Example

$$u(t) = \sin(2\pi/3t) + 1.2 \sin(\pi t + 0.3) + 0.8 \sin(2\pi/5t + 0.4)$$

What's the steady-output of a system with transfer function $G(s)$, if the system has all poles in the left half plane?

Superposition gives:

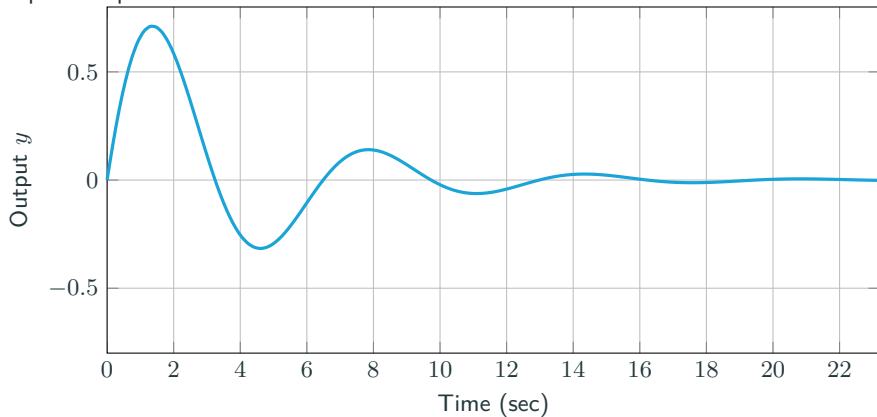
$$\begin{aligned} y(t) = & |G(j2\pi/3)| \sin(2\pi/3t + \angle G(j2\pi/3)) \\ & + 1.2 |G(j\pi)| \sin(\pi t + 0.3 + \angle G(j\pi)) \\ & + 0.8 |G(j2\pi/5)| \sin(2\pi/5t + 0.4 + \angle G(j2\pi/5)) \end{aligned}$$

Example

Consider the simple system

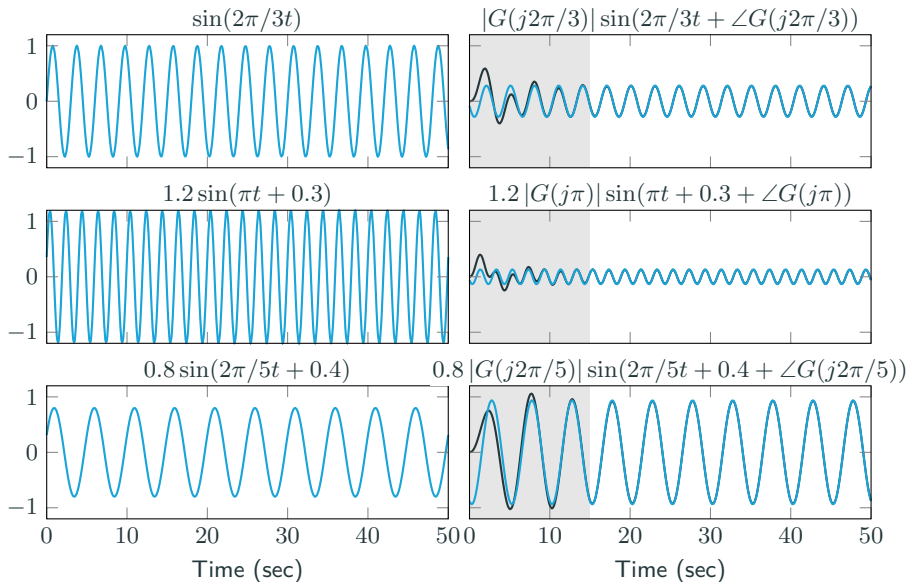
$$G(s) = \frac{1}{s^2 + 0.5s + 1}$$

Impulse response

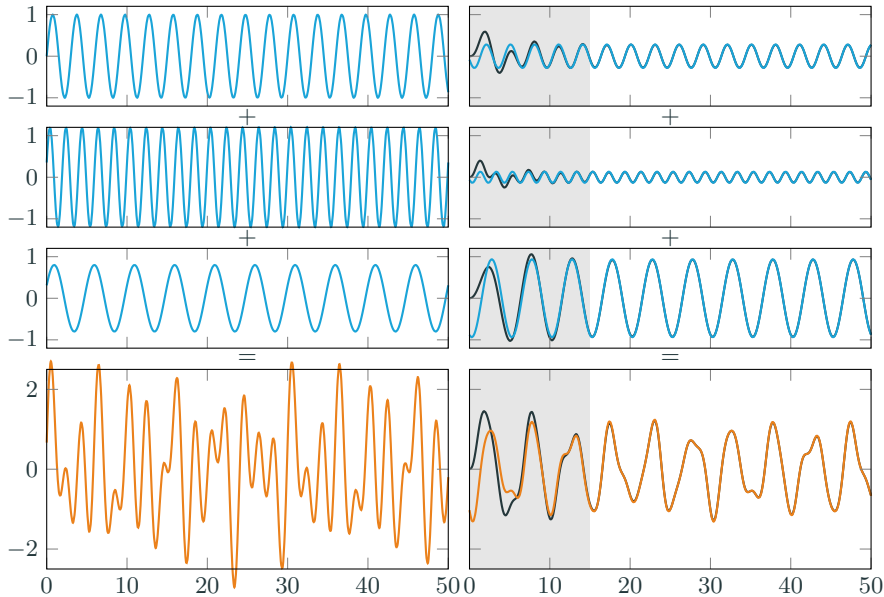


Expect the transient phase to die out around 15 seconds.

Example

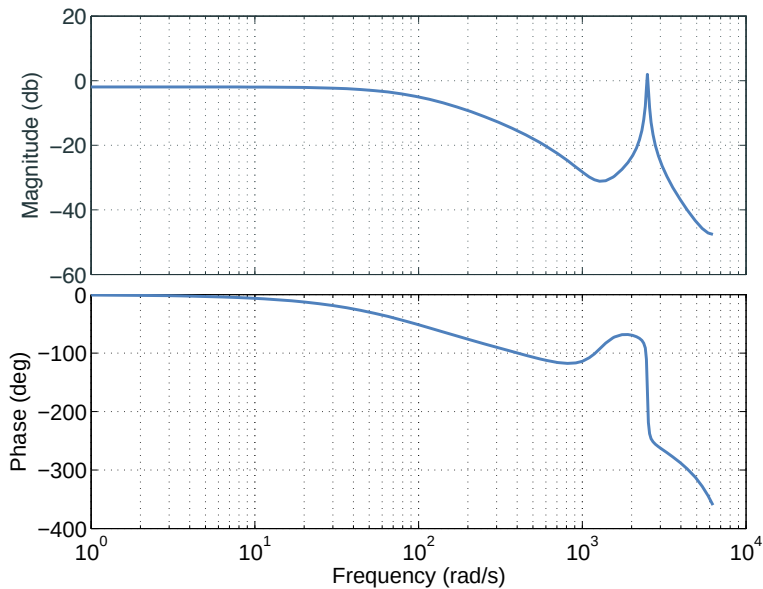


Example



Visualizing the Frequency Response

Visualization: Bode Plot



1. Magnitude plot $|G(j\omega)|$

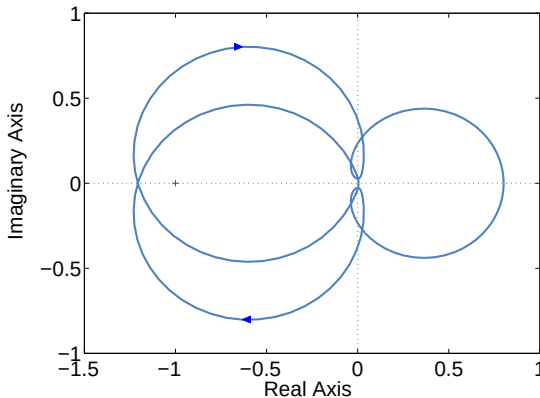
- Plotted in decibels $20 \log_{10}(|G(j\omega)|)$
- X-axis is frequency, usually rad/sec, but sometimes Hz
- Value above 0 $\rightarrow$ the output is larger than the input
- Value below 0 $\rightarrow$ the output is smaller than the input
- All physical systems will tend to $-\infty$ decibels as $\omega \rightarrow \infty$

2. Phase plot $\angle G(j\omega)$

- Generally shown in degrees
- X-axis is frequency, usually rad/sec, but sometimes Hz
- Value above 0 $\rightarrow$ phase advance
- Value below 0 $\rightarrow$ phase lag

- Easy to generate directly from data
- Much can be said from a 'glance'
 - Gain at specific frequencies obvious
 - Resonance frequencies
 - Bandwidth
 - Stability in closed-loop
 - etc
- Commonly used for control design
- Control objectives commonly described using Bode plot
- Can be generalized to multi-input / multi-output systems
- 'Easy' to sketch manually

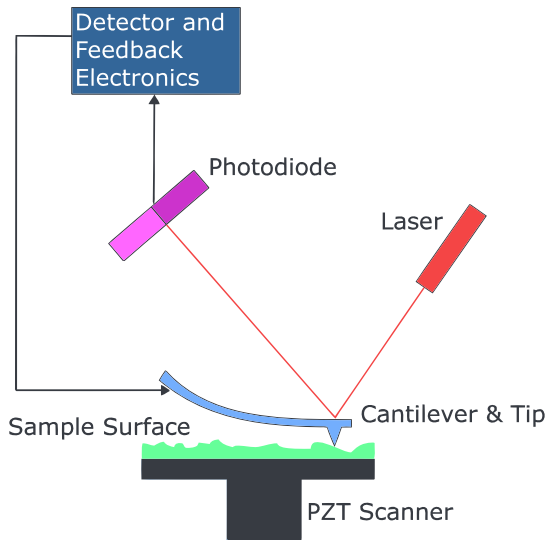
Nyquist Diagram



- Generally used for more theoretical analysis
 - e.g., robust stability, robust performance, etc
- Significantly more complex to draw
 - We could spend weeks learning this...
 - I'll just say "Use a computer" - `nyquist(G)`

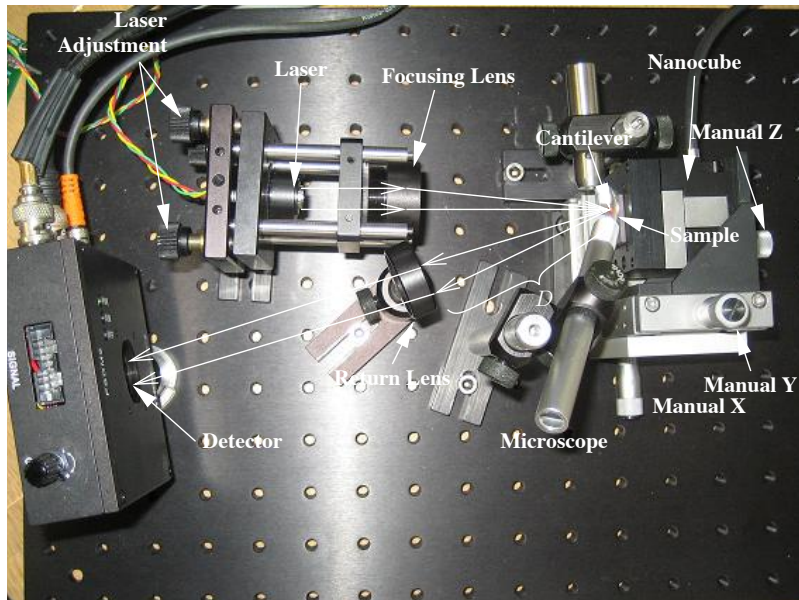
Example: Atomic Force Microscope

Atomic Force Microscope



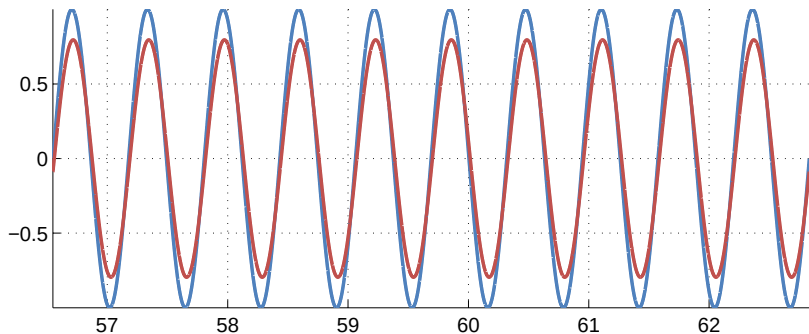
¹Image: Wikipedia

Atomic Force Microscope: Student Experiment



Generating a Bode Plot: Method 1

Drive the system with the input $u(t) = \sin(\omega_i t)$ for several ω_i 's



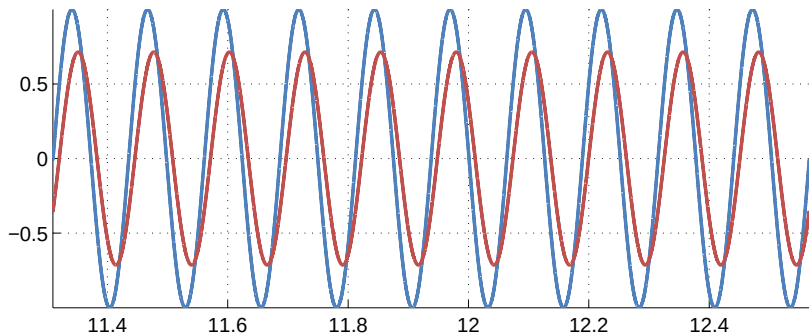
$$\omega = 2\pi 10$$

$$|G(j\omega)| = 7.96 \cdot 10^{-1}$$

$$\angle G(j\omega) = -3.03$$

Generating a Bode Plot: Method 1

Drive the system with the input $u(t) = \sin(\omega_i t)$ for several ω_i 's



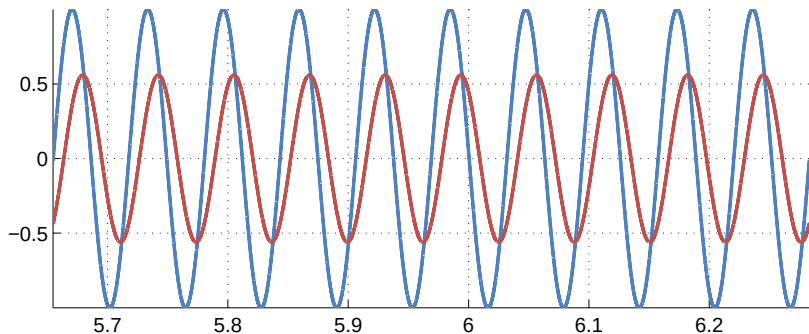
$$\omega = 2\pi 50$$

$$|G(j\omega)| = 7.14 \cdot 10^{-1}$$

$$\angle G(j\omega) = -2.63$$

Generating a Bode Plot: Method 1

Drive the system with the input $u(t) = \sin(\omega_i t)$ for several ω_i 's



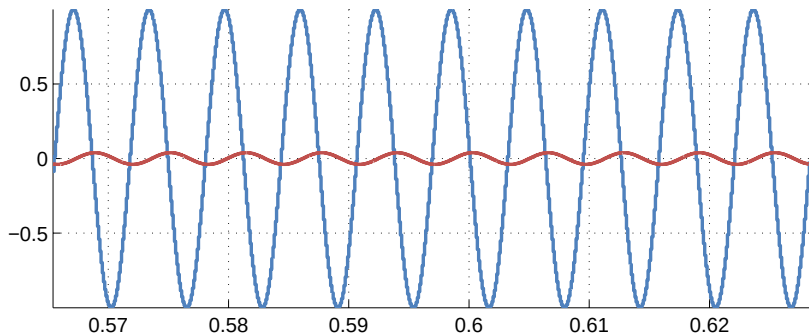
$$\omega = 2\pi 100$$

$$|G(j\omega)| = 5.61 \cdot 10^{-1}$$

$$\angle G(j\omega) = -2.26$$

Generating a Bode Plot: Method 1

Drive the system with the input $u(t) = \sin(\omega_i t)$ for several ω_i 's



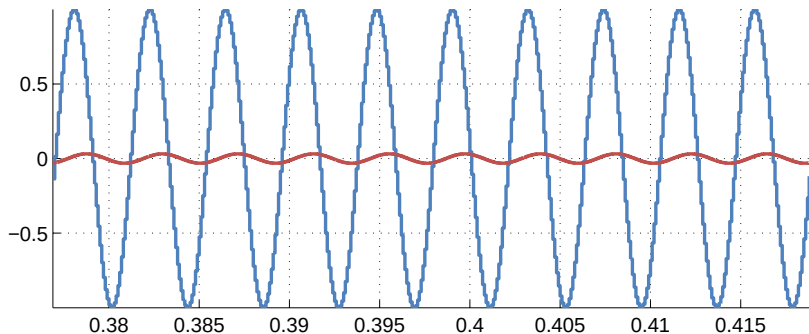
$$\omega = 2\pi 1000$$

$$|G(j\omega)| = 3.92 \cdot 10^{-2}$$

$$\angle G(j\omega) = -1.35$$

Generating a Bode Plot: Method 1

Drive the system with the input $u(t) = \sin(\omega_i t)$ for several ω_i 's



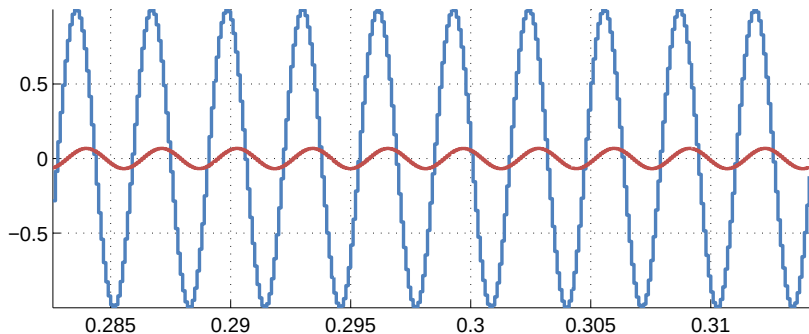
$$\omega = 2\pi 1500$$

$$|G(j\omega)| = 3.16 \cdot 10^{-2}$$

$$\angle G(j\omega) = -2.11$$

Generating a Bode Plot: Method 1

Drive the system with the input $u(t) = \sin(\omega_i t)$ for several ω_i 's



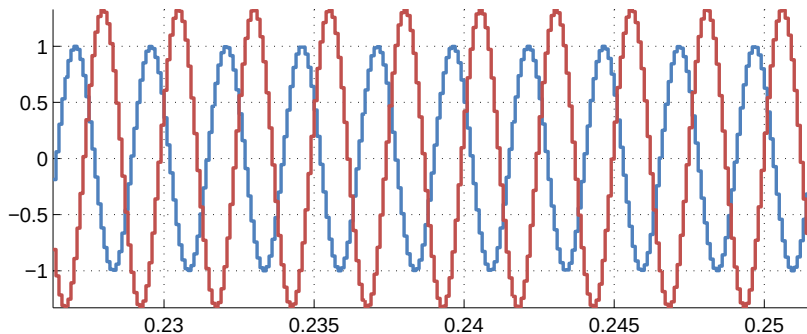
$$\omega = 2\pi 2000$$

$$|G(j\omega)| = 6.80 \cdot 10^{-2}$$

$$\angle G(j\omega) = -2.33$$

Generating a Bode Plot: Method 1

Drive the system with the input $u(t) = \sin(\omega_i t)$ for several ω_i 's



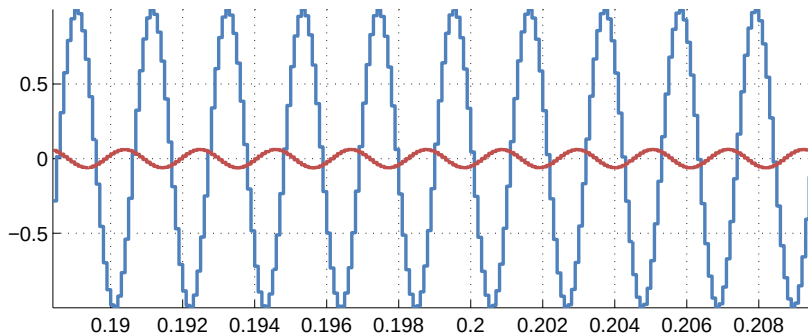
$$\omega = 2\pi 2498$$

$$|G(j\omega)| = 1.33$$

$$\angle G(j\omega) = -0.85$$

Generating a Bode Plot: Method 1

Drive the system with the input $u(t) = \sin(\omega_i t)$ for several ω_i 's



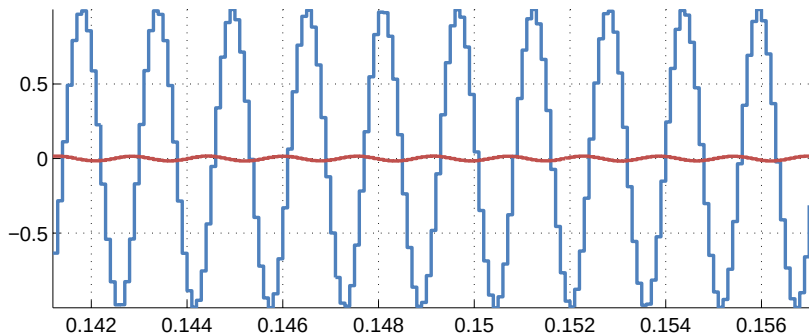
$$\omega = 2\pi 3000$$

$$|G(j\omega)| = 6.16 \cdot 10^{-2}$$

$$\angle G(j\omega) = -0.84$$

Generating a Bode Plot: Method 1

Drive the system with the input $u(t) = \sin(\omega_i t)$ for several ω_i 's



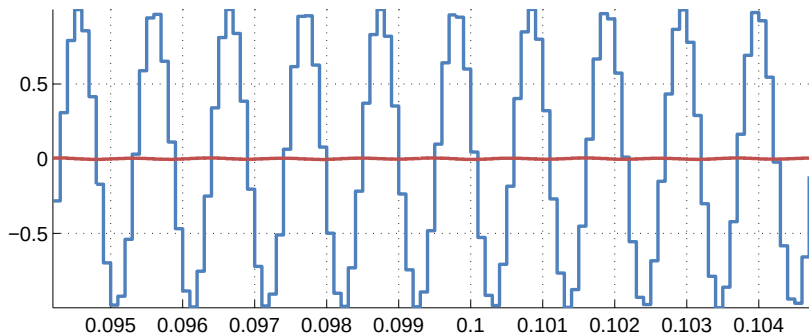
$$\omega = 2\pi 4000$$

$$|G(j\omega)| = 1.67 \cdot 10^{-2}$$

$$\angle G(j\omega) = -1.07$$

Generating a Bode Plot: Method 1

Drive the system with the input $u(t) = \sin(\omega_i t)$ for several ω_i 's



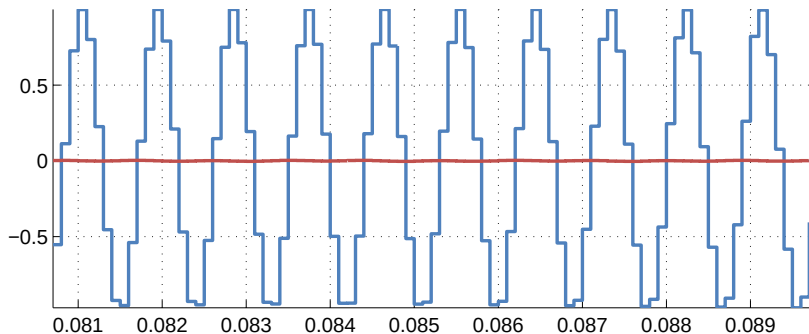
$$\omega = 2\pi 6000$$

$$|G(j\omega)| = 5.25 \cdot 10^{-3}$$

$$\angle G(j\omega) = -1.40$$

Generating a Bode Plot: Method 1

Drive the system with the input $u(t) = \sin(\omega_i t)$ for several ω_i 's



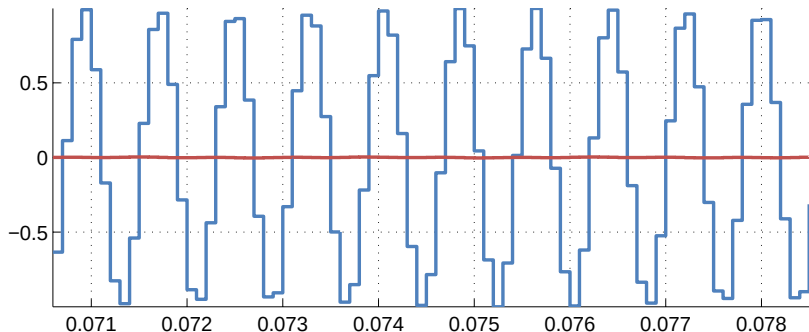
$$\omega = 2\pi 7000$$

$$|G(j\omega)| = 3.54 \cdot 10^{-3}$$

$$\angle G(j\omega) = -1.50$$

Generating a Bode Plot: Method 1

Drive the system with the input $u(t) = \sin(\omega_i t)$ for several ω_i 's

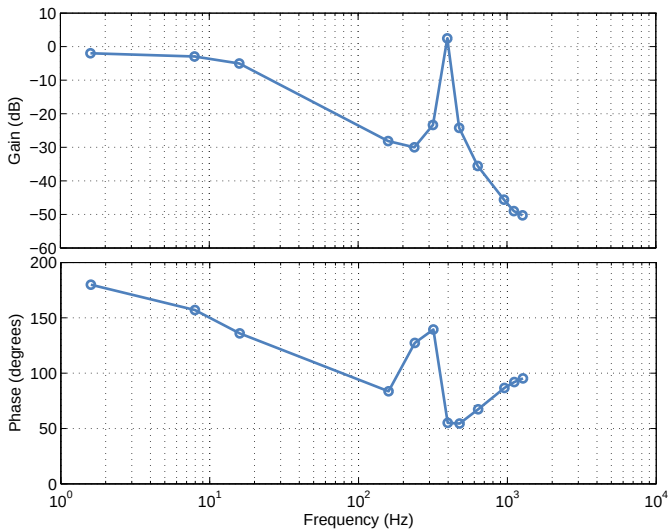


$$\omega = 2\pi 8000$$

$$|G(j\omega)| = 3.06 \cdot 10^{-3}$$

$$\angle G(j\omega) = -1.55$$

Put Sampled Points on Plot



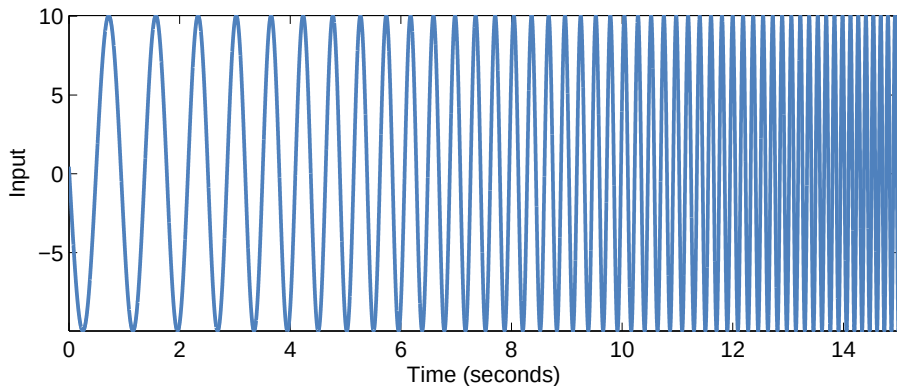
Problem: Need a *lot* of points to get a good fit

Better: Frequency Sweep

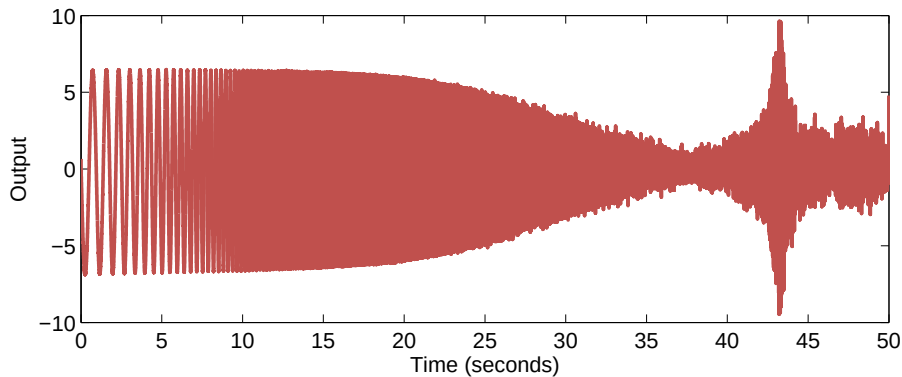
Sweep from f_0 Hz to f_f Hz in t_s seconds

$$u(t) = \cos(\phi(t))$$

$$\frac{d\phi(t)}{dt} = f_0 \left(\frac{f_0}{f_f} \right)^{t/t_s}$$



Output in Response to the Frequency Sweep

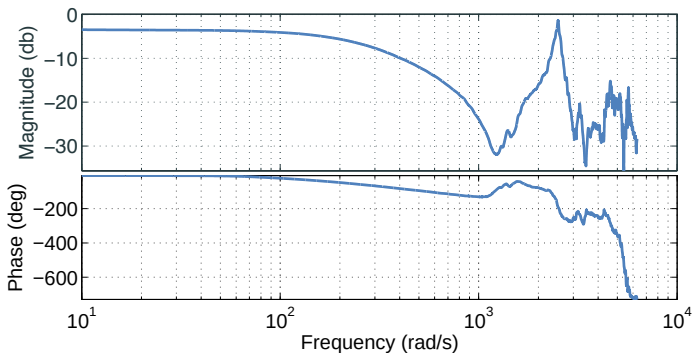


Obtaining the Frequency Response Experimentally

Recall the definition of frequency response:

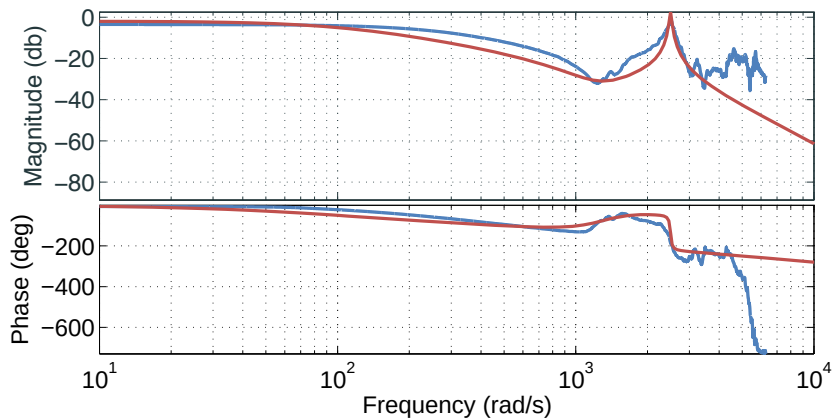
$$G(j\omega) = \frac{Y(j\omega)}{U(j\omega)}$$

and note that $Y(j\omega)$ is the discrete-time Fourier transform of $y(k)$. We just compute the Fourier transform of $y(k)$ and $u(k)$, and take their difference



Interpreting the Frequency Response

$$G(s) = \frac{8.88 \cdot 10^8 (s^2 + 780s + 1.69 \cdot 10^6)}{(s + 3000)(s + 1000)(s + 100)(s^2 + 50s + 6.25 \cdot 10^6)}$$

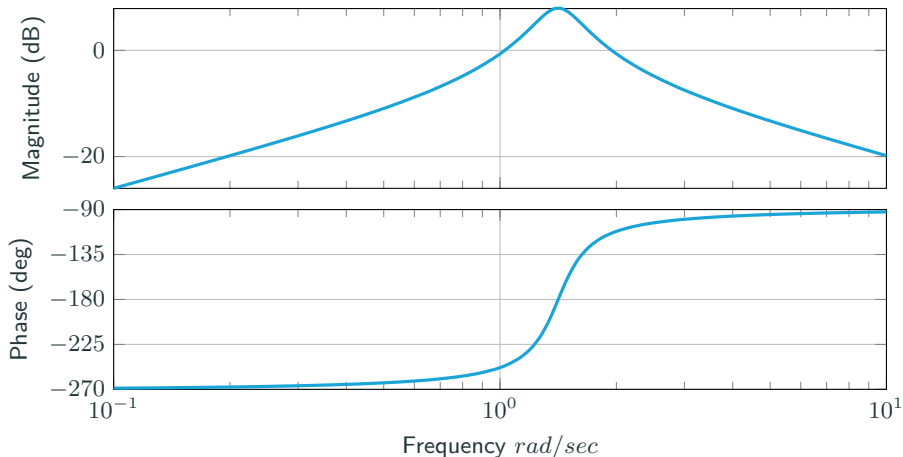


Recall: Sketching Bode Plots

Matlab Commands

The computer way: Grid ω , compute and plot.

```
sys = tf([1 0],[1 -0.4 2]); % Define system  
bode(sys); % Show bode plot
```



Sketching Ratio of Polynomials on a Log-Scale

Sketch Bode plot of continuous-time system $G(s)$

$$G(s) = c \frac{(s + z_1)(s + z_2)}{s(s + p_1)(s + p_2)}$$

Goal: Sketch $G(j\omega)$

For any ω we have a complex number in the form

$$G(j\omega) = \frac{s_1 s_2}{s_3 s_4 s_5} = \frac{r_1 e^{j\theta_1} r_2 e^{j\theta_2}}{r_3 e^{j\theta_3} r_4 e^{j\theta_4} r_5 e^{j\theta_5}} = \frac{r_1 r_2}{r_3 r_4 r_5} e^{j(\theta_1 + \theta_2 - \theta_3 - \theta_4 - \theta_5)}$$

So the magnitude is linear in a log-scale, and the phase is linear

$$\begin{aligned} 20 \log_{10} |G(j\omega)| &= 20 \log_{10} r_1 + 20 \log_{10} r_2 - 20 \log_{10} r_3 - 20 \log_{10} r_4 - 20 \log_{10} r_5 \\ \angle G(j\omega) &= \theta_1 + \theta_2 - \theta_3 - \theta_4 - \theta_5 \end{aligned}$$

Write in Bode Form

Convert to Bode form

$$G(j\omega) = c_0(j\omega)^n \frac{(j\omega\tau_1 + 1)(j\omega\tau_2 + 1) \cdots}{(j\omega\tau_a + 1)(j\omega\tau_b + 1) \cdots}$$

Example:

$$G(s) = 15s^{-1} \frac{(\tau_1 s + 1)(\tau_2 s + 1)}{(\tau_3 s + 1)(\tau_4 s + 1)}$$

The magnitude then of the form

$$\begin{aligned} 20 \log_{10} |G(j\omega)| &= 20 \log_{10} 15 + 20 \log_{10} |j\omega\tau_1 + 1| + 20 \log_{10} |j\omega\tau_2 + 1| \\ &\quad - 20 \log_{10} |j\omega\tau_3 + 1| - 20 \log_{10} |j\omega\tau_4 + 1| \end{aligned}$$

All transfer functions are made of three types of terms

1. $K_0(j\omega)^n$

2. $(j\omega\tau + 1)^{\pm 1}$

3. $\left[\left(\frac{j\omega}{\omega_n} \right)^2 + 2\zeta \frac{j\omega}{\omega_n} + 1 \right]^{\pm 1}$

Magnitude

$$20 \log_{10} |K_0(j\omega)^n| = 20 \log_{10} |K_0| + n 20 \log_{10} |j\omega|$$

This is a straight line with slope of $n \times (20 \text{ decibels per decade})$.

Phase

$$\angle K_0(j\omega)^n = n \times 90^\circ$$

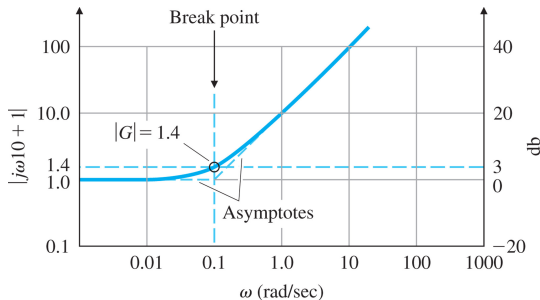
The phase is constant everywhere. Each integrator drops the phase by 90° , and each zero at zero increases it by 90°

Simple pole / zero $(j\omega\tau + 1)^{\pm 1}$

Magnitude $20 \log_{10} |j\omega\tau + 1|$

- $\omega\tau \ll 1 \rightarrow j\omega\tau + 1 \cong 1$
- $\omega\tau \gg 1 \rightarrow j\omega\tau + 1 \cong j\omega\tau$

Example: $j\omega 10 + 1$



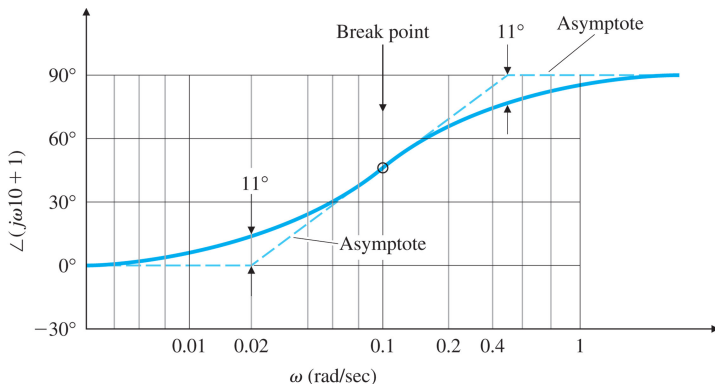
Copyright © 2015 Pearson Education, All Rights Reserved

Simple pole / zero $(j\omega\tau + 1)^{\pm 1}$

Phase $\angle(j\omega\tau + 1)$

- $\omega\tau \ll 1 \rightarrow \angle 1 = 0^\circ$
- $\omega\tau \approx 1 \rightarrow \angle(j\omega\tau + 1) \approx 45^\circ$
- $\omega\tau \gg 1 \rightarrow \angle j\omega\tau = 90^\circ$

Example: $j\omega 10 + 1$



Second-order term $\left[\left(\frac{j\omega}{\omega_n} \right)^2 + 2\zeta \frac{j\omega}{\omega_n} + 1 \right]^{\pm 1}$

Magnitude

Very similar to first-order term, with breakpoint at $\omega = \omega_n$

- $\frac{\omega}{\omega_n} \ll 1 \quad \rightarrow \quad \left(\frac{j\omega}{\omega_n} \right)^2 + 2\zeta \frac{j\omega}{\omega_n} + 1 \cong 1$
- $\frac{\omega}{\omega_n} \gg 1 \quad \rightarrow \quad \left(\frac{j\omega}{\omega_n} \right)^2 + 2\zeta \frac{j\omega}{\omega_n} + 1 \cong \left(\frac{j\omega}{\omega_n} \right)^2$

Second-order term $\left[\left(\frac{j\omega}{\omega_n} \right)^2 + 2\zeta \frac{j\omega}{\omega_n} + 1 \right]^{\pm 1}$

Magnitude

Very similar to first-order term, with breakpoint at $\omega = \omega_n$

- $\frac{\omega}{\omega_n} \ll 1 \quad \rightarrow \quad \left(\frac{j\omega}{\omega_n} \right)^2 + 2\zeta \frac{j\omega}{\omega_n} + 1 \cong 1$
- $\frac{\omega}{\omega_n} \gg 1 \quad \rightarrow \quad \left(\frac{j\omega}{\omega_n} \right)^2 + 2\zeta \frac{j\omega}{\omega_n} + 1 \cong \left(\frac{j\omega}{\omega_n} \right)^2$

The magnitude around the crossover frequency is impacted by the damping ratio

$$\begin{aligned} |G(j\omega_n)| &= \left| \left(\frac{j\omega_n}{\omega_n} \right)^2 + 2\zeta \frac{j\omega_n}{\omega_n} + 1 \right|^n \\ &= |-1 + 2\zeta j + 1|^n \\ &= |2\zeta|^n \end{aligned}$$

So for the most common case of $n = -1$ (a resonant pole) we have

$$|G(j\omega_n)| \cong \frac{1}{2\zeta}$$

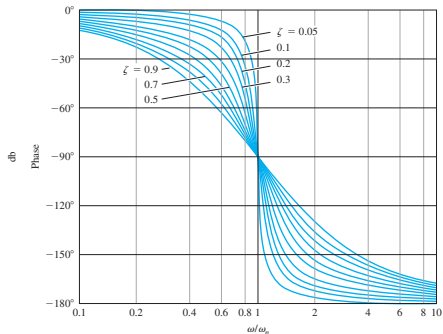
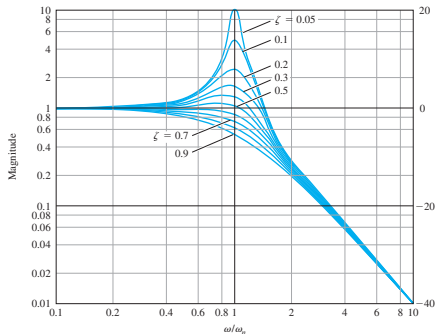
Phase

$$G(j\omega) = \left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1$$

- $\omega \ll \omega_n \quad \rightarrow \quad \angle 1 = 0^\circ$
- $\omega \cong \omega_n \quad \rightarrow \quad \angle j^2 + 2\zeta j + 1 = 90^\circ$
- $\omega \gg \omega_n \quad \rightarrow \quad \angle \left(\frac{j\omega}{\omega_n}\right)^2 = 180^\circ$

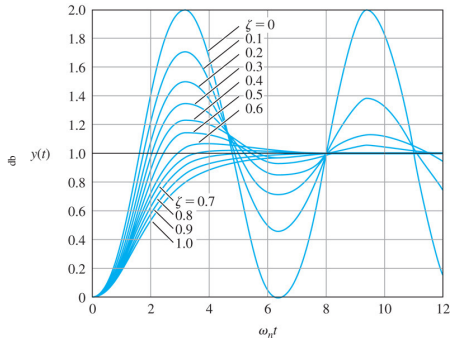
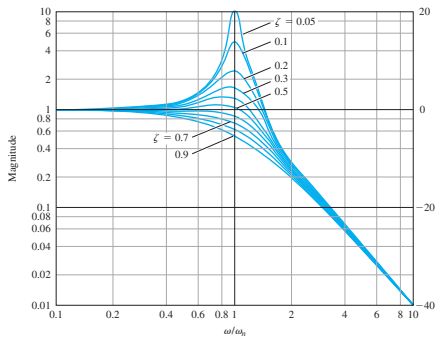
Second-Order Poles / Zeros

$$G(j\omega) = \frac{1}{(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1}$$



Second-Order Poles / Zeros

$$G(w) = \frac{1}{(w/\omega_n)^2 + 2\zeta(w/\omega_n) + 1}$$



■ Damping ζ

- transient-response overshoot (approx $1/2\zeta$ for $\zeta < 0.5$)
- peak in frequency response magnitude

■ Natural frequency ω_n

- approximately equal to bandwidth
- proportional to the rise time

Unstable Poles and Non-minimum Phase Zeros

What happens if the pole/zero is in the right half plane?

$$G_1(j\omega) = j\omega\tau + 1$$

$$G_2(j\omega) = j\omega\tau - 1$$

Magnitude

$$|G_1(j\omega)| = |G_2(j\omega)|$$

Magnitude is the same

Phase

- $\omega\tau \ll 1 \quad \rightarrow \quad \angle G_1(j\omega) = 0^\circ \quad \angle G_2(j\omega) = 180^\circ$
- $\omega\tau \cong 1 \quad \rightarrow \quad \angle G_1(j\omega) \cong 45^\circ \quad \angle G_2(j\omega) \cong 135^\circ$
- $\omega\tau \gg 1 \quad \rightarrow \quad \angle G_1(j\omega) = 90^\circ \quad \angle G_2(j\omega) = 90^\circ$

Phase goes in the opposite direction for RHP poles / zeros

Magnitude

- Each zero increases the slope by 20dB/dec
- Each pole decreases the slope by 20dB/dec
- Complex poles/zeros have a resonant peak; larger with lower damping ratio
- Physical systems must have a negative slope as $\omega \rightarrow \infty$
- Slope changes occur at pole/zero locations

Phase

- Negative zero $\rightarrow 90^\circ$, positive zero $\rightarrow -90^\circ$
- Negative pole $\rightarrow -90^\circ$, positive pole $\rightarrow 90^\circ$
- Physical systems must have a negative phase as $\omega \rightarrow \infty$
- Phase changes begin/end $\sim 1/2$ decade before/after poles/zeros

Three Indicative Examples

$$1. \ G(s) = \frac{2000(s + 0.5)}{s(s + 10)(s + 50)}$$

$$2. \ G(s) = \frac{10}{s(s^2 + 0.4s + 4)}$$

$$3. \ G(s) = \frac{s - 1}{s + 1}$$

Detailed summary of plots on Moodle (and in exercises)

Summary

The steady-state output of a linear system in response to a sinusoid is a sinusoid of the same frequency with a phase shift of:

- $\angle G(j\omega)$ and a magnitude of $|G(j\omega)|$

Why important?

- Very useful method to experimentally capture the dynamics of a system
- Common control objectives expressed in terms of frequency requirements
- Can determine closed-loop behaviour, from open-loop frequency response

In coming weeks we will use the Bode plot of the open-loop system to:

- Compute key metrics defining the robustness of the system
- Shape the response of closed-loop system by 'modifying' the bode plot with a controller